

Section 11.9 Representation of functions as power series

Goal: represent certain types of functions as sums of power series. This is usually useful in integration, solving differential equations, approximations of functions in computers and calculators, etc...

We have seen one such representation already:

$$\boxed{1} \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\dots, \text{ whenever } |x| < 1 \quad (\text{See Section 11.8})$$

Ex ① Express the function $\frac{1}{1+2x}$ as the sum of a power series, and find the radius and interval of convergence.

- If we replace " x " by " $-2x$ " in Eqn $\boxed{1}$ we obtain

$$\frac{1}{1-(-2x)} = \frac{1}{1+2x} = \sum_{n=0}^{\infty} (-2x)^n = \sum_{n=0}^{\infty} (-2)^n x^n = 1-2x+4x^2-8x^3+\dots, \text{ when } |-2x| < 1$$

So, the power series of $f(x) = \frac{1}{1+2x}$ is $\sum_{n=0}^{\infty} (-2)^n x^n$. Since this representation holds for $|-2x| < 1$, i.e. $|x| < \frac{1}{2}$, we get that the radius of conv. is $R = \frac{1}{2}$, and the interval of convergence is $(-\frac{1}{2}, \frac{1}{2})$.

Ex ② Express $\frac{10}{1+x^2}$ as the sum of a power series, and find the radius and interval of convergence. Replace " x " by " $-x^2$ " in Eqn $\boxed{1}$, then multiply by 10; we get $\frac{10}{1+x^2} = 10 \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} 10(-1)^n x^{2n} = 10-10x^2+10x^4-10x^6+\dots$

The series converges when $|-x^2| < 1$, or $|x| < 1$; i.e. on $-1 < x < 1$. So, $R=1$, and the I.O.C is $(-1, 1)$.

Ex ③ Express $f(x) = \frac{x^2}{1-\frac{x}{2}}$ as the sum of a power series. What is the I.O.C?

$$\frac{x^2}{1-\frac{x}{2}} = x^2 \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2^n} x^{n+2} = x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \dots \text{ Series conv}$$

when $|\frac{x}{2}| < 1 \Rightarrow |x| < 2$. So, $R=2$, and the I.O.C is $(-2, 2)$.

Differentiation and integration of power Series.

Theorem: suppose the power series $\sum_{n=0}^{\infty} C_n (x-a)^n$ has radius of conv. $R > 0$, then the function f defined by $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1 (x-a) + C_2 (x-a)^2 + \dots$ is differentiable (and therefore continuous) on the I.O.C. $(a-R, a+R)$, and

- $f'(x) = C_1 + 2C_2 (x-a) + 3C_3 (x-a)^2 + \dots = \sum_{n=1}^{\infty} n \cdot C_n (x-a)^{n-1}$
- $\int f(x) dx = C + C_0 (x-a) + C_1 \frac{(x-a)^2}{2} + C_2 \frac{(x-a)^3}{3} + \dots = C + \sum_{n=0}^{\infty} C_n \frac{(x-a)^{n+1}}{n+1}$.

This means we can differentiate and integrate power series term-by-term. Note: the 2 "New" power series ($f'(x)$ and $\int f(x) dx$) in (i) and (ii) have the same radius of conv. (R) as $f(x)$. However, these new series may not converge at the endpoints of the I.O.C.

Ex ① Express $f(x) = \frac{1}{(1-x)^2}$ as the sum of a power series. what is the I.O.C.?

Observe that $\frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{1}{(1-x)^2}$. Let us then differentiate Eqn ① w.r.t. x :

$$\frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{1}{(1-x)^2} = \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \dots, |x| < 1.$$

Ex ② Express $f(x) = \ln(1+x)$ as the sum of a power series. what is the I.O.C.?

First, we have $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n, |x| < 1$ (or $|x| < 1$).

Observe now that $\int \frac{1}{1+x} dx = \ln(1+x)$. Therefore,

$$\ln(1+x) = \int \sum_{n=0}^{\infty} (-1)^n x^n dx = \sum_{n=0}^{\infty} \int (-1)^n x^n dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C, |x| < 1.$$

What is the integration constant C ? let's plug $x=0$ in the equality above:

$$\ln(1+0) = \sum_{n=0}^{\infty} (-1)^n \frac{0^{n+1}}{n+1} + C = 0 + C \Rightarrow C=0. \text{ Finally we get}$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \text{ whenever } |x| < 1.$$

Series also conv. at $x=1$ (Alt. Series Test) $\Rightarrow \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots = \sum \frac{(-1)^n}{n} !!!$